The relationship (4.2) should be added to (6.2). We then obtain a system of equations and boundary conditions to determine u_0, w_0, T_1, T_2, M_1, M_2. It is often convenient to integrate this system directly in the form presented here. In some cases it is expedient to eliminate the static factors by means of (4.2). Consequently, we obtain a system of two equations in u_0, w_0. It is expedient to use it if purely geometric conditions are given. On the basis of the boundary value problems obtained, the influence of the lack of shallowness and of the physical nonlinearity on the critical state of the shell can be investigated in particular.

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Translated by M. D. F.

UDC 539.3

AN ENERGY INEQUALITY IN THE THEORY OF PLATE BENDING

PMM Vol. 37, №5, 1973, pp. 940-944
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The construction of error estimates of approximate theories of plates is based on inequalities relating the three-dimensional elastic energy of the plate and the elastic energy according to a two-dimensional approximate theory.

The inequality
$$E_0(u_\alpha) \leqslant E(w, w_\alpha) \tag{1}$$

has been proved in [1] in the case of extension of an isotropic homogeneous linearly elastic plate of constant thickness h . Here E is the three-dimensional elastic energy, w_α are the tangential components of the displacement vector, w is the displacement along the normal to the plate (*), E_0 is the elastic energy by the theory of the plane

*) For the extension w_α is an even and w is an odd function of the transverse coordinate x, while for bending w_α is an odd, and w an even function of x.

state of stress, and u_α are the tangential displacements averaged over the thickness

$$u_\alpha = \frac{1}{h}\int_{-h/2}^{h/2} w_\alpha\, dx$$

The asymptotic accuracy of the theory of the plane state of stress is proved in [1] for $h \to 0$ by using (1). Naturally it would be expected that an analogous inequality

$$E_K(u) \leqslant E(w, w_\alpha) \tag{2}$$

holds for plate bending, where E_K is the elastic energy by Kirchhoff theory, and u is the transverse displacement averaged over the thickness. However, inequality (2) is incorrect (*). This is easily seen by setting $w_\alpha = 0,\ w = u(x^\alpha)$, for example ($x^\alpha$ are the coordinates in the plate middle plane). Then the quadratic form $\partial^2 u/\partial x^\alpha \partial x^\beta$ is in the left side of the inequality, and $\partial u/\partial x^\alpha$ in the right. As is known, it is impossible to find the upper bound of the second derivatives of functions in terms of the first in the norm L_2. It turns out that the elastic energy E_R of the Reissner model yields the exact upper bound of the three-dimensional elastic energy of a bent plate

$$E_R(u, \psi_\alpha) \leqslant E(w, w_\alpha) \tag{3}$$

where

$$u = \int_{-h/2}^{h/2} \left(x^2 - \frac{h^2}{4}\right) w\, dx \Big/ \int_{-h/2}^{h/2} \left(x^2 - \frac{h^2}{4}\right) dx$$

$$\psi_\alpha = \int_{-h/2}^{h/2} x w_\alpha\, dx \Big/ \int_{-h/2}^{h/2} x^2 dx \tag{4}$$

This paper is devoted to a proof of the inequality (3) and some estimates resulting from this inequality.

Let Ω denote the middle plane of the plate, Γ the boundary of Ω, V the domain occupied by the plate

$$V = \{x,\ x^\alpha : x^\alpha \in \Omega,\ \ -h/2 \leqslant x \leqslant h/2\}$$

and U the internal energy per unit volume of the plate

$$E = \int_V U dx^1\, dx^2\, dx$$

The internal energy U for a linearly elastic isotropic homogeneous plate is defined by the formula

$$2U = \lambda(\varepsilon_\alpha{}^\alpha)^2 + 2\mu\varepsilon_{\alpha\beta}\varepsilon^{\alpha\beta} + 2\lambda\varepsilon\varepsilon_\alpha{}^\alpha + (\lambda + 2\mu)\varepsilon^2 + 4\mu\varepsilon_\alpha\varepsilon^\alpha \tag{5}$$

$$\varepsilon_{\alpha\beta} = w_{(\alpha,\beta)}, \qquad \varepsilon = \frac{\partial w}{\partial x}, \qquad 2\varepsilon_\alpha = w_{,\alpha} + \frac{\partial w_\alpha}{\partial x}$$

Here the comma before the Greek subscripts denotes differentiation with respect to x^α, and the parentheses in the subscripts is the symmetrization operation.

The elasticity energy of the Reissner model is given by the relations

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*) For this reason, D. Morgenstern apparently had to apply a method of proof different from that which he used in the theory of the plane state of stress [1] in order to prove the asymptotic accuracy of the Kirchhoff hypotheses [2].

$$E_R = \int_\Omega \Phi dx^1 dx^2$$

$$2\Phi = \frac{h^3}{12} 2\mu \left[\frac{\lambda}{\lambda + 2\mu} (\psi^{\sigma}_{,\sigma})^2 + \psi_{(\alpha,\beta)}\psi^{(\alpha,\beta)}\right] + \frac{5}{6}\mu h (u_{,\alpha} + \psi_\alpha)(u^{,\alpha} + \psi^\alpha)$$

The estimate of the elastic energy (3) for continuously differentiable functions w, w^α results from the following inequalities:

1°. Let a_α denote the quantities

$$a_\alpha = \int_{-h/2}^{h/2} \varepsilon_\alpha \left(x^2 - \frac{h^2}{4}\right) dx$$

Then the Cauchy-Buniakowski inequality

$$a_\alpha a^\alpha \leqslant \int_{-h/2}^{h/2} \varepsilon_\alpha \varepsilon^\alpha dx \int_{-h/2}^{h/2} \left(x^2 - \frac{h^2}{4}\right)^2 dx \equiv \frac{h^5}{30} \int_{-h/2}^{h/2} \varepsilon_\alpha \varepsilon^\alpha dx \tag{6}$$

Evidently

$$a_\alpha = \frac{1}{2} \int_{-h/2}^{h/2} \left(\frac{\partial w}{\partial x^\alpha} + \frac{\partial w_\alpha}{\partial x}\right)\left(x^2 - \frac{h^2}{4}\right) dx = -\frac{h^3}{12}(u_{,\alpha} + \psi_\alpha) \tag{7}$$

Substituting (7) into (6), we obtain

$$\frac{5}{6} h (u_{,\alpha} + \psi_\alpha)(u^{,\alpha} + \psi^\alpha) \leqslant 4 \int_{-h/2}^{h/2} \varepsilon_\alpha \varepsilon^\alpha dx \tag{8}$$

2°. From the Cauchy-Buniakowski inequality, the definition of ψ^α from (4) and the definition of $\varepsilon_{\alpha\beta}$ from (5), we have

$$\frac{h^3}{12} (\psi^{\alpha}_{,\alpha})^2 \leqslant \int_{-h/2}^{h/2} (\varepsilon_\alpha{}^\alpha)^2 dx, \qquad \frac{h^3}{12} \psi_{(\alpha,\beta)}\psi^{(\alpha,\beta)} \leqslant \int_{-h/2}^{h/2} \varepsilon_{\alpha\beta}\varepsilon^{\alpha\beta} dx \tag{9}$$

3°. The inequality

$$-\frac{\lambda^2}{\lambda + 2\mu} (\varepsilon_\alpha{}^\alpha)^2 \leqslant 2\lambda \varepsilon_\alpha{}^\alpha \varepsilon + (\lambda + 2\mu)\varepsilon^2 \tag{10}$$

is evident. Replacing the terms $2\lambda\varepsilon_\alpha{}^\alpha\varepsilon + (\lambda + 2\mu)\varepsilon^2$ by the smaller quantity $(\varepsilon_\alpha{}^\alpha)^2\lambda^2 / (\lambda + 2\mu)$ in the internal energy U, integrating over the domain V and substituting $\varepsilon_{\alpha\beta}\varepsilon^{\alpha\beta}$ and $\varepsilon_\alpha\varepsilon^\alpha$ which do not exceed their magnitude, in place of the integrals of $(\varepsilon_\alpha{}^\alpha)^2$ with respect to x according to (8) and (9), we arrive at the inequality (3). The inequality (3) is exact. The equality sign holds in (3) for example, for functions of the form ($u(x^\alpha)$ is an arbitrary harmonic function):

$$w = u(x^\alpha), \quad w_\alpha = -u_{,\alpha} x \quad \left(\frac{\partial^2 u}{\partial x_1{}^2} + \frac{\partial^2 u}{\partial x_2{}^2} = 0\right)$$

The enrgy estimate of the error in the approximate Reissner theory can be constructed by using the inequality (3). In order not to make the exposition of the technical details unwieldy, let us consider the volume forces to be absent, and let us take the simplest boundary conditions: zero surface forces for $x = \pm h/2$, and on the edge of the plate $S = \{x, x^\alpha : x^\alpha \in \Gamma, -h/2 \leqslant x \leqslant h/2\}$ the displacements are given as

$$w|_S = u_\Gamma + \frac{1}{2}\chi_\Gamma\left(x^2 - \frac{h^2}{20}\right), \quad w^\alpha|_S = \psi_\Gamma{}^\alpha x + \frac{1}{3}\theta_\Gamma{}^\alpha x\left(x^2 - \frac{3h^2}{20}\right) \tag{11}$$

where u_Γ, χ_Γ, $\psi_\Gamma{}^\alpha$, $\theta_\Gamma{}^\alpha$ are known functions on the contour Γ.

The exact solution of the problem is given by the minimizing the element w^*, $w_\alpha{}^*$ of the functional E in the set of functions w, w_α which satisfy the boundary conditions (11), and the approximate solution by minimizing the element u^{**}, $\psi_\alpha{}^{**}$ of the functional E_R in the set of functions u, ψ_α satisfying the boundary conditions

$$u|_\Gamma = u_\Gamma, \quad \psi^\alpha|_\Gamma = \psi_\Gamma{}^\alpha \tag{12}$$

It is assumed that the minimizing functions are continuously differentiable. We find the error in the Reissner theory by the method of two-sided estimates. This method has been used in plate theory in [1 – 3]. The method of two-sided estimates consists of establishing inequalities of the form

$$E_R(u^{**}, \psi_\alpha{}^{**}) \leqslant E_R(u^*, \psi_\alpha{}^*) \leqslant E(w^*, w_\alpha{}^*) \leqslant E_R(u^{**}, \psi_\alpha{}^{**}) + F \tag{13}$$

where u^* and $\psi_\alpha{}^*$ are defined as functions of w^*, $w_\alpha{}^*$ by (4). By virtue of the identity

$$E_R(u^{**} - u, \psi_\alpha{}^{**} - \psi_\alpha) = E_R(u, \psi_\alpha) - E_R(u^{**}, \psi_\alpha{}^{**})$$

which holds for any functions u, ψ_α taking the same values on Γ as do u^{**}, $\psi_\alpha{}^{**}$, the desired estimate

$$E_R(u^{**} - u^*, \psi_\alpha{}^{**} - \psi_\alpha{}^*) \leqslant F \tag{14}$$

results from (13). Therefore, there remains to prove (13) and to obtain an expression for F. The second inequality in (13) results from the inequality (3) proved above, and the first from the fact that u^* and $\psi_\alpha{}^*$ belong to the set of functions by which the functional E_R is minimized, and $E_R(u^{**}, \psi_\alpha{}^{**})$ is its minimal value. To construct the upper bound $E(w^*, w_\alpha{}^*)$, let us consider the value of the functional E in displacement fields of the form

$$w = u + \frac{1}{2}\chi\left(x^2 - \frac{h^2}{20}\right) \tag{15}$$

$$w_\alpha = \psi_\alpha x + \frac{1}{3}\theta_\alpha x\left(x^2 - \frac{3h^2}{20}\right)$$

We obtain [4]

$$E = E_R(u, \psi_\alpha) + F(u, \psi_\alpha, \chi, \theta_\alpha) \tag{16}$$

$$F = \frac{h^3}{24}\int_\Omega \Big\{(\lambda + 2\mu)\left(\chi + \frac{\lambda}{\lambda + 2\mu}\psi^\sigma_{,\sigma}\right)^2 + \frac{2\mu h^2}{25}\left[\theta_\alpha + \frac{5}{h^2}\left(u_{,\alpha} + \psi_\alpha + \frac{h^2}{10}\chi_{,\alpha}\right)\right]^2 + \frac{h^4}{2100}\left[\lambda(\theta^\sigma_{,\sigma})^2 + 2\mu\theta_{(\alpha,\beta)}\theta^{(\alpha,\beta)}\right]\Big\}\, dx^1\, dx^2$$

The function u corresponds to the function u^* in [4].

Let us set $u = u^{**}$, $\psi_\alpha = \psi_\alpha{}^{**}$ in (15), (16) and let us still keep the functions χ and θ_α arbitrary, however, for the displacement field (15) to satisfy the boundary conditions (11), we subject χ and θ_α to the constraints

$$\chi|_\Gamma = \chi_\Gamma, \quad \theta^\alpha|_\Gamma = \theta_\Gamma{}^\alpha \tag{17}$$

Then, since the displacement field (15) belongs to the set of functions in which the functional E is minimized because of the mentioned selection of u, χ, ψ_α, θ_α, and $E(w^*, w_\alpha{}^*)$ is its minimum value, the third inequality of (13) and the estimate (14) hold. The functional $F(u^{**}, \psi_\alpha{}^{**}, \chi, \theta_\alpha)$ in (13), (14) is defined by (16). The left side in (14) is independent of χ and θ_α, hence any functions satisfying the boundary condition (17), including functions achieving the minimum of the functional F, can be taken as χ and θ_α. In the particular case when χ_Γ and $\theta_\Gamma{}^\alpha$ in the boundary conditions (11) are selected so that the equalities

$$\chi_\Gamma = -\frac{\lambda}{\lambda+2\mu}\psi^{**\sigma}_{,\sigma}|_\Gamma \quad (18)$$
$$\theta_{\alpha\Gamma} = -\frac{5}{h^2}\left(u^{**}_{,\alpha} + \psi_\alpha{}^{**} - \frac{h^2}{10}\frac{\lambda}{\lambda+2\mu}\psi^{**\sigma}_{,\sigma\alpha}\right)\Big|_\Gamma$$

are valid, the functions

$$\chi = -\frac{\lambda}{\lambda+2\mu}\psi^{**\sigma}_{,\sigma}$$
$$\theta_\alpha = -\frac{5}{h^2}\left(u^{**}_{,\alpha} + \psi_\alpha{}^{**} - \frac{h^2}{10}\frac{\lambda}{\lambda+2\mu}\psi^{**\sigma}_{,\sigma\alpha}\right)$$

can be taken as χ and θ_α. The functional F hence becomes

$$F = \frac{h^7}{50\,400}\int\limits_\Omega (\lambda(\theta^\sigma_{,\sigma})^2 + 2\mu\theta_{(\alpha,\beta)}\theta^{(\alpha,\beta)})\,dx^1\,dx^2$$

It can be shown that the functional F is of the order of h^6 for u_Γ independent of the parameter h and $|\psi_\Gamma{}^\alpha\tau_\alpha + du_\Gamma/ds| \leqslant ch^2$ (τ_α is the vector tangent to Γ). However, in the general case when (18) are not true for χ_Γ and $\theta_\Gamma{}^\alpha$ because of the edge effect, the functional F is of the order of h^4.

It follows from the estimates obtained that the Reissner model is asymptotically exact in the energy norm. Since the Reissner model goes over into the Kirchhoff model as $h \to 0$, the asymptotic accuracy of the Kirchhoff theory also follows from these estimates.

In conclusion, let us note that the inequality (3) and the corresponding error estimates of the Reissner theory are extended by word-for-word duplication to homogeneous anisotropic plates of constant thickness which have a plane of elastic symmetry parallel to the middle plane at each point. For such plates the elastic energy per unit volume is

$$2U = A^{\alpha\beta\gamma\delta}\varepsilon_{\alpha\beta}\varepsilon_{\gamma\delta} + 2A^{\alpha\beta}\varepsilon_{\alpha\beta}\varepsilon + A\varepsilon^2 + 4G^{\alpha\beta}\varepsilon_\alpha\varepsilon_\beta$$

The inequalities

$$\frac{5}{6}hG^{\alpha\beta}(u_{,\alpha}+\psi_\alpha)(u_{,\beta}+\psi_\beta) \leqslant 4\int\limits_{-h/2}^{h/2} G^{\alpha\beta}\varepsilon_\alpha\varepsilon_\beta\,dx \quad (19)$$
$$\frac{h^3}{12}E^{\alpha\beta\gamma\delta}\psi_{(\alpha,\beta)}\psi_{(\gamma,\delta)} \leqslant \int\limits_{-h/2}^{h/2} E^{\alpha\beta\gamma\delta}\varepsilon_{\alpha\beta}\varepsilon_{\gamma\delta}\,dx,$$
$$E^{\alpha\beta\gamma\delta} = A^{\alpha\beta\gamma\delta} - A^{-1}A^{\alpha\beta}A^{\gamma\delta}$$
$$-A^{-1}(A^{\alpha\beta}\varepsilon_{\alpha\beta})^2 \leqslant A^{\alpha\beta}\varepsilon_{\alpha\beta}\varepsilon + A\varepsilon^2$$

must be used in place of inequalities (8) – (10). It follows from the inequalities (19) that an exact lower bound for the elastic energy of a bent anisotropic plate is given by the elastic energy E_R defined by the formula

$$E_R = \frac{1}{2}\int\limits_\Omega\left[\frac{h^3}{12}E^{\alpha\beta\gamma\delta}\psi_{(\alpha,\beta)}\psi_{(\gamma,\delta)} + \frac{5}{6}hG^{\alpha\beta}(u_{,\alpha}+\psi_\alpha)(u_{,\beta}+\psi_\beta)\right]dx^1dx^2$$

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Translated by M. D. F.

UDC 539.3

APPLICATION OF HANKEL TRANSFORMS TO THE SOLUTION OF AXISYMMETRIC PROBLEMS WHEN THE MODULUS OF ELASTICITY IS A POWER FUNCTION OF DEPTH

PMM Vol. 37, №5, 1973, pp. 945-948
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(Received October 9, 1972)

The application of Hankel transforms to the three-dimensional axisymmetric problems of the theory of elasticity, in the case when the modulus of elasticity is a power function of depth, leads to a system of ordinary differential equations [1] whose solution presents some mathematical difficulty. Therefore, in [1, 2] the solution of these problems has been carried out by applying transformations expounded in [3].

In the sequel we construct the fundamental system of solutions of the ordinary differential equations mentioned above and we give the solution for two boundary value problems in the case of very special conditions.

1. In the case of axial symmetry, the displacement equations of the theory of elasticity have the form

$$(\lambda+2\mu)\left[\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}-\frac{u}{r^2}\right]+(\lambda+\mu)\frac{\partial^2 w}{\partial r\partial z}+\mu\frac{\partial^2 u}{\partial z^2}+\frac{\partial\mu}{\partial z}\left[\frac{\partial u}{\partial z}+\frac{\partial w}{\partial r}\right]=0$$

$$\mu\left[\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}\right]+(\lambda+\mu)\left[\frac{\partial^2 u}{\partial r\partial z}+\frac{1}{r}\frac{\partial u}{\partial z}\right]+(\lambda+2\mu)\frac{\partial^2 w}{\partial z^2}-\frac{\partial\lambda}{\partial z}\left[\frac{\partial u}{\partial z}+\frac{u}{r}\right]+\frac{\partial(2\mu+1)}{\partial z}\frac{\partial w}{\partial z}=0$$

We apply the Fourier method, by using the Hankel's transforms in the following form

$$u(r,z)=\int_0^\infty \varphi(s,z)J_1(sr)\,ds,\quad w(r,z)=\int_0^\infty f(s,z)J_0(sr)\,ds$$